Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Monday 3 June 2019

Morning (Time: 1 hour 30 minutes) | Paper Reference **9FM0/01**

Further Mathematics

Advanced

Paper 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

Answer ALL questions. Write your answers in the spaces provided.

1.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(4)

(b) find the values of a, b, c and d.

(5)

(a) remembering the 4 options for the complex roots of a quartic equation according to the Fundamental Law of Algebra:

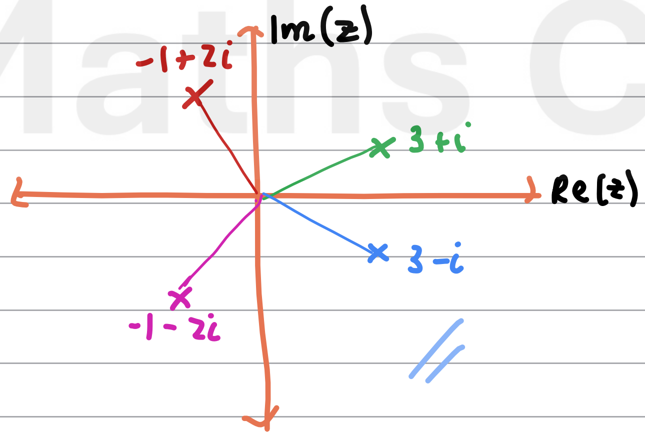
- 4 real roots
- 2 real roots, 2 complex roots (a complex conjugate pair)
- 4 complex roots (2 complex conjugate pairs)

in the question we are given two complex roots that aren't a complex conjugate pair ∴ know we're looking at the third option

following rule that if $z = a+bi$, $z^* = a-bi$

$z_1 = -1+2i$, $z^* = -1-2i$
$z_2 = 3-i$, $z^* = 3+i$ } ∴ forming our four complex roots

↳ need to plot these on an Argand diagram - if in the form $a+bi$, plot as Cartesian coordinates (a,b)



(b) to find corresponding quartic equation to given roots - 2 main methods

METHOD 1: use complex conjugate roots to find two quadratic factors of polynomial of degree 4

...first forming quadratics out of given complex conjugate pairs:

Question 1 continued

WAY 1: using fact that general equation for quadratic is $z^2-(\Sigma\alpha)+\alpha\beta$

$z_{1,1^*}$: $\boxed{\Sigma\alpha} = -1+2i+-1-2i$
$= -2$
(or using memorised that
$z+z^* = 2a = 2(-1)$)

$\boxed{\alpha\beta} = (-1+2i)(-1-2i)$
$= 1+2i-2i-4i^2$
$= 5$
(or memorised $zz^* = a^2+b^2$)
subbing into general quadratic formula
$\Rightarrow z^2+2z+5$

$z_{2,2^*}$: $\boxed{\Sigma\alpha} = (3-i)(3+i)$
(or memorised that $z+z^* = 2(a)$)
$= 2(3)$

$\boxed{\alpha\beta} = (3-i)(3+i)$
$= 9+3i-3i-i^2$
$= 9-(-1)$
$= 10$
(or using memorised $zz^* = a^2+b^2$
$= (3)^2+(-1)^2$
$= 10$

$\Rightarrow z^2-6z+10$

WAY 2: using factor theorem - if $x-z$ is a factor then $f(z)=0$

$f(z) = (z-(-1+2i))(z-(-1-2i))(z-(3-i))(z-(3+i))$
expand brackets
$= (z+1-2i)(z+1+2i)(z-3+i)(z-3-i)$
expand 2 quadratics
$= (z^2+(1+2i)z+(1-2i)z+5)(z^2+(-3-i)z+(-3+i)z+10)$
$= (z^2+2z+5)(z^2-6z+10)$

now we have these two quadratics - have to expand these to get quartic:

$$
\begin{aligned}
(z^2+2z+5)(z^2-6z+10) &= z^4-6z^3+10z^2 \\
&\quad +2z^3-12z^2+20z \\
&\quad +5z^2-30z+50 \\
&= z^4-4z^3+3z^2-10z+50
\end{aligned}
$$

$\boxed{\Rightarrow a=-4,\ b=3,\ c=-10,\ d=50}$

METHOD 2: using roots of polynomial equations

let $\alpha=-1+2i$, $\beta=-1-2i$, $\gamma=3-i$, $\delta=3+i$

looking to apply roots of polynomials formulae to given quartic

$f(z) = z^4 + \underset{\Sigma\alpha}{a}z^3 + \underset{\Sigma\alpha\beta}{b}z^2 + \underset{\Sigma\alpha\beta\gamma}{c}z + \underset{\alpha\beta\gamma\delta}{d}$

where sum of roots $= \Sigma\alpha = -a/1 = -a$

Question 1 continued

sum of product pairs $= \Sigma\alpha\beta = b/1 = b$

sum of product triples $= \Sigma\alpha\beta\gamma = -c/1 = -c$

product of roots $= \alpha\beta\gamma\delta = d/1 = d$

now working above out using our given roots:

$\boxed{\Sigma\alpha} = -1+2i + -1-2i + 3-i + 3+i$

$= -2+6$

$= 4$

equate to above

$\Rightarrow 4 = -a$

$\Rightarrow a = -4$

$\boxed{\Sigma\alpha\beta} = (-1+2i)(-1-2i) + (-1+2i)(3-i) + (-1+2i)(3+i) + (-1-2i)(3-i) + (-1-2i)(3+i) + (3-i)(3+i)$

on calc or manually

$5 + (-1+7i) + (-5+5i) + (-5-5i) + (-1-7i) + 10$

real and imaginary

$(5-1\cancel{-5}\cancel{-5}-1+10) + i(7+5-5-7)i$

$= 3$

equate to formulae results

$\Rightarrow 3 = b$

$\boxed{\Sigma\alpha\beta\gamma} = (-1+2i)(-1-2i)(3-i) + (-1+2i)(-1-2i)(3+i) + (-1-2i)(3-i)(3+i) + (-1+2i)(3-i)(3+i)$

$= 15 - \cancel{5i} + 15 + \cancel{5i} - 10 - \cancel{20i} - 10 + \cancel{20i}$

$= 30 - 20$

$= 10$

equate to formulae results

$10 = -c$

$\Rightarrow c = -10$

$\boxed{\alpha\beta\gamma\delta} = (-1+2i)(-1-2i)(3-i)(3+i)$

$= 5(10)$

$= 50$

equating to formulae results

$50 = d$

$\boxed{\Rightarrow a = -4,\ b = 3,\ c = -10,\ d = 50}$

Question 1 continued

(Total for Question 1 is 9 marks)

2. Show that

$$\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,dx = \ln k$$

where k is a rational number to be found.

(7)

notice we're asked to integrate a fractional expression - looking at ways to integrate a fractional expression

Fractional expressions

4a. Can I **split the numerator**?
Is there a single term in the denominator?
4b. Can I do **partial fractions**?
Does the denominator factorise?
4c. Can I do **algebraic division**?
Is the fraction improper?

need to do it by partial fractions (steps explained more in detail on pg.9 - end of question)

may be tempted to write:

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{A}{2x^2+3} + \frac{B}{x+1}$$

but in FM - need most general expression in the numerator

need $\dfrac{8x-12}{(2x^2+3)(x+1)} = \dfrac{Ax+B}{2x^2+3} + \dfrac{C}{x+1}$

$\Rightarrow 8x-12 = (Ax+B)(x+1) + C(2x^2+3)$

WAY 1: compare coefficients

... x^2 :

$0 = A + 2C$ - ①

... x :

$8 = A + B$ - ②

... constants:

$-12 = B + 3C$ - ③

eg. manipulate ② to get A the subject

$A = 8 - B$

sub into ①

$0 = 8 - B + 2C$

$\Rightarrow B - 2C = 8$ - ④

and solve ③ and ④ simultaneously - using calc equation solver / by elimination

$B + 3C = -12$
$-\ \ B - 2C = 8$
$5C = -20$

WAY 2: by substitution

let $x = -1$,

$-20 = 5C$
$\div 5 \quad \div 5$
$\Rightarrow \boxed{C = -4}$

rest by comparing coefficients

... x^2 :

$0 = A + 2C$
$0 = A + 2(-4)$
$0 = A - 8$
$\Rightarrow \boxed{A = 8}$

... x :

$8 = A + B$
$8 = 8 + B$
$\Rightarrow \boxed{B = 0}$

$\div 5 \quad \div 5$

$\Rightarrow \boxed{C = -4}$

Question 2 continued

subbing 'C' into ①

$0 = A + 2(-4)$

$0 = A - 8$

$\boxed{\Rightarrow A = 8}$

subbing into ③

$8 = 8 + B$

$\boxed{\Rightarrow B = 0}$

$\Rightarrow \dfrac{8x - 12}{(2x^2+3)(x+1)} = \dfrac{8x}{2x^2+3} - \dfrac{4}{x+1}$

now indefinite integral of partial fractions

$$\int \frac{8x}{2x^2+3} - \frac{4}{x+1}\,dx = \int \frac{8x}{2x^2+3}\,dx - 4\int \frac{1}{x+1}\,dx$$

notice how both integrals can be evaluated using reverse chain rule

consider: $\ln(2x^2+3)$

differentiate: $\dfrac{1}{2x^2+3} \times 4x$

$= \dfrac{4x}{2x^2+3}$

but need $8x$ in numerator:

scale: $\times 2$

$\boxed{= 2\ln(2x^2+3) + c}$

now for next,

consider : $\ln(x+1)$

differentiate: $\dfrac{1}{x+1}$

scale: $\times 1$

$\boxed{= \ln(x+1)}$

subbing into indefinite integral:

$2\ln(2x^2+3) - 4\ln(x+1) + c$

and manipulate using log laws - power rule and quotient rule

Question 2 continued $= \ln\left(\frac{(2x^2+3)^2}{(x+1)^4}\right)$

now applying limit

$$\lim_{t\to\infty}\left[\ln\left(\frac{(2x^2+3)^2}{(x+1)^4}\right)\right]_0^t$$

$$\Rightarrow \lim_{t\to\infty}\left\{\left[\ln\frac{(2t^2+3)^2}{(t+1)^4}\right]-\left[\ln\left(\frac{9}{1}\right)\right]\right\}$$

$$= \lim_{t\to\infty}\left\{\left[\ln\frac{(4t^4+12t^2+9)}{t^4+4t^3+6t^2+4t+1}\right]-\left[\ln 9\right]\right\}$$

and using L'hospital rule ($\div t^4$)

$$\lim_{t\to\infty}\left(\ln\left(\frac{4+\frac{12}{t^2}+\frac{9}{t^4}}{1+\frac{4}{t}+\frac{6}{t^2}+\frac{4}{t^3}+\frac{1}{t^4}}\right)-\ln(9)\right)$$

as $t\to\infty$,

$$\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)} \to \ln\left(\frac{4}{1}\right)-\ln(9)$$

which using log quotient rule:

$$=\ln\left(\frac{4}{9}\right)$$

$$\Rightarrow \boxed{k=4/9}$$

DO NOT WRITE IN THIS AREA

Question 2 continued

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ *means* $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)} dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of-1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)} dx = \boxed{\ln f(x)} + C$$

   Method: copy $\ln$(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x) f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $$\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3} \, dx \text{ etc}$$
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}} dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}} dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2} dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(Total for Question 2 is 7 marks)

3.

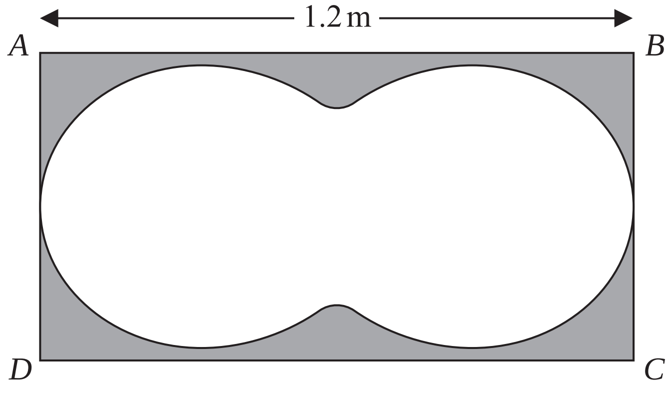


Diagram not to scale

Figure 1

Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m. The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$ **(2)**

Hence, given that $AD = 60$ cm,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures. **(8)**

(a) notice how r_{max} gives HALF the length of tabletop

so need to maximise

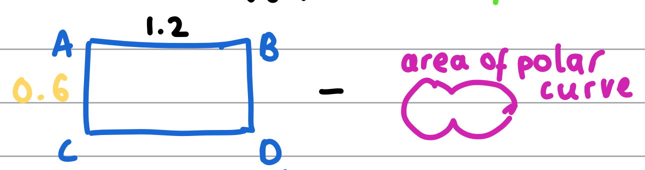
$\underbrace{r = 0.4 + a\cos 2\theta}$ and equate to $\frac{1.2}{2} = 0.6$

at max. when $\cos 2\theta = 1$

$\Rightarrow 0.4 + a = 0.6$

$\Rightarrow \boxed{a = 0.2}$

(b) main strategy for area of wooden i.e shaded:



area of rectangle

Question 3 continued

...first area of rectangle:

1.2

0.6

$= 1.2 \times 0.6$

$= 0.72 m^2$

...now area of polar curve:

using formula for polar integration:

$\pi \int r^2 d\theta$

subbing in our 'r' with part (a)'s

$a = 0.2$ AND exploiting symmetry:

$$2 \times \cancel{\tfrac{1}{2}} \int_0^{\pi} (0.4 + 0.2\cos 2\theta)^2 \, d\theta$$

expanding inside integral

$$\int (0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta) d\theta$$

but can't really integrate trig powers ∴ use memorised

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\int_0^{\pi} 0.16 + 0.16\cos 2\theta + 0.04\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \int_0^{\pi} (0.18 + 0.16\cos 2\theta + 0.02\cos 4\theta) d\theta$$

integrate

$$= \left[0.18\theta + 0.08\sin 2\theta + 0.005\sin 4\theta\right]_0^{\pi}$$

$$= \left\{\left[0.18(\pi) + 0.08\sin(2\pi) + 0.05(\sin(4\pi))\right] - 0\right\}$$

$$= 0.18\pi = \frac{9}{50}\pi$$

∴ area of wooden $= 0.72 - 0.18\pi$

$$\boxed{= 0.155 m^2}$$

Question 3 continued

(Total for Question 3 is 10 marks)

4. Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

notice we have the summation of a fractional expression ∴ cannot exploit the Yr 1 summation formulae - only method left is method of differences (Yr 2)

first rewriting u_r in terms of partial fractions to get something in form:

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{n} f(r+1)$$

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

WAY 1: compare coefficients:

...r^2:

$0 = A + B + C$ - ①

...r:

$0 = 5A + 4B + 3C$ - ②

...constants:

$1 = 6A + 3B + 2C$ - ③

make C the subject of ①

$C = -A - B$

and sub into ②

$0 = 5A + 4B + 3(-A-B)$

$\Rightarrow 0 = 2A + B$ - ④

and the same into ③

$1 = 6A + 3B + 2(-A-B)$

$\Rightarrow 1 = 4A + B$ - ⑤

solving ④ and ⑤ simultaneously

⑤-④ $4A + B = 1$

$-\ 2A + B = 0$

$2A = 1$

$\div 2 \quad \div 2$

$\Rightarrow \boxed{A = \frac{1}{2}}$

WAY 2: by substitution

let $r = -2$,

$1 = A(0)(1) + B(-1)(1) + C(-1)(0)$

$\Rightarrow 1 = -B$

$\Rightarrow \boxed{B = 1}$

let $r = -3$,

$1 = A(-1)(0) + B(-2)(0) + C(-2)(-1)$

$\Rightarrow 1 = 2C$

$\div 2 \quad \div 2$

$\Rightarrow \boxed{C = \frac{1}{2}}$

let $r = -1$,

$1 = A(1)(2) + B(0)(2) + C(0)(1)$

$1 = 2A$

$\div 2 \quad \div 2$

$\Rightarrow \boxed{A = \frac{1}{2}}$

Question 4 continued

subbing into ④

$$0 = 2(1/2) + B$$

$$\Rightarrow \boxed{B = -1}$$

and finally into highlighted rearranged

$$c = -1/2 - (-1)$$

$$\boxed{c = 1/2}$$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \sum_{r=0}^{n} \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

evaluate above for $r = 0, 1, 2, \ldots, n-2, n-1, n$

... two main ways:

WAY 1: numerical method

$$u_0: \frac{1}{2(0+1)} - \frac{1}{0+2} + \frac{1}{2(0+3)}$$

$$\boxed{= \frac{1}{2} - \frac{1}{2} + \cancel{\frac{1}{6}}}$$

$$u_1: \frac{1}{2(1+1)} - \frac{1}{1+2} + \frac{1}{2(1+3)}$$

$$\boxed{= \frac{1}{4} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{8}}}$$

$$u_2: \frac{1}{2(2+1)} - \frac{1}{2+2} + \frac{1}{2(2+3)}$$

$$\boxed{= \cancel{\frac{1}{6}} - \frac{1}{4} + \frac{1}{10}}$$

$$u_3: \frac{1}{2(3+1)} - \frac{1}{3+2} + \frac{1}{2(3+3)}$$

$$\boxed{= \cancel{\frac{1}{8}} - \frac{1}{5} + \frac{1}{12}}$$

⋮

$$u_{n-2}: \frac{1}{2(n-2+1)} - \frac{1}{n-2+2} + \frac{1}{2(n-2+3)}$$

$$\boxed{= \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}}$$

$$u_{n-1}: \frac{1}{2(n-1+1)} - \frac{1}{n-1+2} + \frac{1}{2(n-1+3)}$$

$$\boxed{= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}}$$

$u_n: \boxed{\frac{1}{2(n+1)} - \frac{1}{n+2} + \frac{1}{2(n+3)}}$

notice this unusual 'L' shape left after cancelling, left with:

$$\frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{n+2} + \frac{1}{2(n+3)}$$

collecting like terms

$$= \frac{1}{4} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$$

WAY 2: mechanical method

asked to evaluate

$$\sum_{r=0}^{n} \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

taking $\frac{1}{2}$ out

$$\frac{1}{2} \sum_{r=0}^{n} \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{(r+3)}$$

let $f(r) = \frac{1}{r+1}$, $f(r+1) = \frac{1}{r+2}$, $f(r+2) = \frac{1}{r+3}$

so rewriting above as

$$\frac{1}{2} \sum_{r=1}^{n} f(r) - 2f(r+1) + f(r+2)$$

evaluate at $r = 0, 1, 2, 3 \ldots n-2, n-1, n$

$u_0 : f(0) - 2f(1) + \cancel{f(2)}$

$u_1 : f(1) - 2\cancel{f(2)} + f(3)$

$u_2 : \cancel{f(2)} - 2f(\cancel{3}) + f(4)$

$\vdots$

$u_{n-2} : f(n-2) - 2f(n-1) + \cancel{f(n)}$

$u_{n-1} : f(n-1) - 2f(\cancel{n}) + f(n+1)$

$u_n : f(n) - 2f(n+1) + f(n+2)$

after cancelling left with

$$f(0) - 2f(1) + f(1) + f(n+1) - 2f(n+1) + f(n+2)$$

subbing into function

$$\frac{1}{2}\left(\frac{1}{1+0} - 2\left(\frac{1}{1+1}\right) + \frac{1}{2} + \frac{1}{n+1+1} - 2\left(\frac{1}{n+2}\right) + \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(1 - 1 + \frac{1}{2} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2} + \frac{1}{n+3}\right)$$

expand ½ into bracket

$$= \frac{1}{4} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$$

Question 4 continued

now manipulate to get common denominator

$$\frac{(n+2)(n+3)-2(n+3)+2(n+2)}{4(n+2)(n+3)}$$

expand numerator

$$\frac{n^2+5n+6-2n-6+2n+4}{4(n+2)(n+3)}$$

collect like terms in numerator

$$\frac{n^2+5n+4}{4(n+2)(n+3)}$$

factorise quadratic

$$\boxed{\frac{(n+4)(n+1)}{4(n+2)(n+3)}}$$

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Question 4 continued

(Total for Question 4 is 5 marks)

5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{dS}{dt} = 3 - \frac{2S}{100 + t}$$

(4)

(b) Hence find the number of grams of salt in the tank after 10 minutes.

(5)

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

(3)

(d) Evaluate the model.

(1)

notice this is a 'filling the container' 1ODE problem ∴ need to follow the following format:

volume of pure water in tank after 't' mins: $100 + (3t - 2t)$

$= 100 + t$

grams of salt after t mins $= \frac{S}{100+t}$

rate of salt in: $3 \times 1 = 3$

rate of salt out: $2 \times \frac{S}{100+t}$

$= \frac{2S}{100+t}$

and, by definition: $\frac{dS}{dt}$ = rate in − rate out

$$\boxed{\Rightarrow \frac{dS}{dt} = 3 - \frac{2S}{100+t}}$$

(b) asked to solve 1ODE; getting in form $\frac{dy}{dx} + Py = Q$

P 6 1 1 7 7 A 0 1 8 3 2

Question 5 continued

$$\frac{ds}{dt} + 2\frac{s}{100+t} = 3$$

see straight away that can't use separation of variables as expression involves addition rather than the product of two functions

$\therefore$ checking for reverse product rule :

$\frac{d}{dt}(s) = \frac{ds}{dt}$ ✓

$$\frac{ds}{dt} + 2\frac{s}{100+t} = 3$$

$\frac{d}{dt}(1) \neq \frac{2}{100+t}$ ×

$\therefore$ not reverse product rule

hence hints at need to introduce an integrating factor – $I.F = e^{\int P dx}$

$$= e^{\int \frac{2}{100+t} dt} = e^{2\ln(100+t)} = e^{\ln(100+t)^2}$$

$$= (100+t)^2$$

multiplying through by this I.F

$$(100+t)^2 \frac{ds}{dt} + 2s\frac{\cancel{(100+t)^2}^{\,100+t}}{\cancel{(100+t)}} = 3(100+t)^2$$

$\frac{d}{dt}(s) = \frac{ds}{dt}$ ✓

$$(100+t)^2 \frac{ds}{dt} + 2s(100+t) = 3(100+t)^2$$

$\frac{d}{dt}(100+t)^2 = 2(100+t)$ ✓

now checking for reverse product rule (above)

$\therefore$ can rewrite equation as derivative of product of S and its derivatives

$$\frac{d}{dt}\left(s(100+t)^2\right) = 3(100+t)^2$$

and integrating both sides

$$s(100+t)^2 = \int 3(100+t)^2\, dt$$

$$\Rightarrow s(100+t)^2 = (100+t)^3 + c$$

or expanding RHS

$\boxed{G.S}$: $s(100+t)^2 = 30{,}000t + 300t^2 + t^3 + c$

now finding particular solution using initial conditions =)

when $t=0,\ s=0$

$$0(100+0)^2 = (100+0)^3 + c$$

$$0 = 1{,}000{,}000 + c$$

$$\Rightarrow c = -1{,}000{,}000 \text{ or } -10^6$$

Question 5 continued

P.S: $s(100+t)^2 = (100+t)^3 - 10^6$

but question wants us to get 's' when $t=10$ - subbing this in:

$$s(100+10)^2 = (100+10)^3 - 10^6$$

$$s(110)^2 = (110)^3 - 10^6$$

$$s(12{,}100) = 1{,}331{,}000 - 10^6$$

$$12{,}100s = 331{,}000$$

$\div 12{,}100$ $\div 12{,}100$

$$s = \frac{3310}{121} \text{ (g)}$$

$$= 27.355...$$

$$\boxed{= 27g}$$

(c) METHOD 1: using 's'

when salt reaches 0.9 g/l, the valve at the bottom of the tank must be closed i.e amount of salt, $s = 0.9(100+t)$

↳ subbing into P.S from (a):

$$0.9(100+t) = \frac{(100+t)^3 - 10^6}{(100+t)^2}$$

$\times (100+t)^2$ $\times (100+t)^2$

$$0.9(100+t)^3 = (100+t)^3 - 10^6$$

collect like terms

$$0.1(100+t)^3 = 10^6$$

$\div 0.1$ $\div 0.1$

$$(100+t)^3 = 10^7$$

cube root

$$100 + t = 215.44..$$

$$\boxed{\Rightarrow t = 115 \text{ mins}}$$

METHOD 2: using concentration formula

$$\text{concentration} = \frac{\text{mass}}{\text{volume}}$$

$$\text{mass} = 100 + t - \frac{10^6}{(100+t)^2}$$

$$\text{volume} = 100 + t$$

$$0.9 = 100 + t - \frac{10^6}{(100+t)^2} \Big/ 100+t$$

Question 5 continued

$\times(100+t)$ $\times(100+t)$

$$0.9(100+t) = 100+t - \frac{10^6}{(100+t)^2}$$

$\times(100+t)^2$ $\times(100+t)^2$

$$0.9(100+t)^3 = (100+t)^3 - 10^6$$

collect like terms

$$0.1(100+t)^3 = 10^6$$

$\div 0.1$ $\div 0.1$

$$(100+t)^3 = 10^7$$

cube root

$$(100+t) = 215.44..$$

$$\Rightarrow \boxed{t = 115 \text{ mins}}$$

(d) possible limitations:

- unlikely that mixing is instantaneous
- model only valid when tank not full
- valve closed - model not valid
- unlikely concentration of salt water entering tank will remain exactly the same

(Total for Question 5 is 13 marks)

6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

notice this is a proof by induction i.e proving a conjecture is true for all $n \in \mathbb{Z}^+$; here we have a divisibility proof

step 1: base case : prove true for $n=1$

$$f(1) = 3^{2(1)+4} - 2^{2(1)}$$
$$= 3^6 - 2^2$$
$$= 729 - 4$$
$$= 725 = 145(5)$$

$\therefore$ true for $n=1$

step 2: assumption step: assume true for $n=k$

$f(k) = 3^{2k+4} - 2^{2k}$ which is divisible by 5 for all $k \in \mathbb{N}$

step 3: induction step: prove true for $n=k+1$

$$f(k+1) = 3^{2(k+1)+4} - 2^{2(k+1)}$$
$$= 3^{2k+6} - 2^{2k+2}$$

breaking this up so we get some multiple of $f(k)$

$$= 3^{2k+4}(3^2) - 2^{2k}(2^2)$$
$$= 9(3^{2k+4}) - 4(2^{2k})$$

...two main ways to manipulate above:

WAY 1: splitting above to get a multiple of f(k) and of 5

$$9(3^{2k+4}) - 4(2^{k})$$

5 4

$$5(3^{2k+4}) + 4(3^{2k+4} - 2^{2k})$$

replacing blue highlight with $f(k)$ -assumption step

$$5(3^{2k+4}) + 4(f(k))$$

multiple of 5

assumed divisible by 5

P 6 1 1 7 7 A 0 2 2 3 2

$\therefore$ true for $n=k+1$

Question 6 continued

WAY 2 : $f(k+1)-f(k)$

$$9(3^{2k+4})-4(2^{k})-3^{2k+4}+2^{2k}$$

collect like terms

$$=8(3^{2k+4})-3(2^{2k})$$

now splitting this to get $f(k)$ factorised

$$5(3^{2k+4})+3(3^{2k+4}-2^{2k})$$

replace above with $f(k)$

$$=5(3^{2k+4})+3f(k)$$

$$\Rightarrow f(k+1)=\underbrace{5(3^{2k+4})}_{\text{multiple of } 5}+\underbrace{3f(k)}_{\text{assumed divisible by } 5}$$

$\therefore$ true for $n=k+1$

Step 4: conclusion step:

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $k\in\mathbb{N}$

(Total for Question 6 is 6 marks)

7. The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane.

(3)

(b) Write down a vector equation for the plane containing l_1 and l_2

(1)

(c) Find, to the nearest degree, the acute angle between l_1 and l_2

(3)

(a) WAY 1: point of intersection

notice line ℓ_1 is given in cartesian equation form and ℓ_2 in vector parametric form so converting ℓ_1 into vector parametric

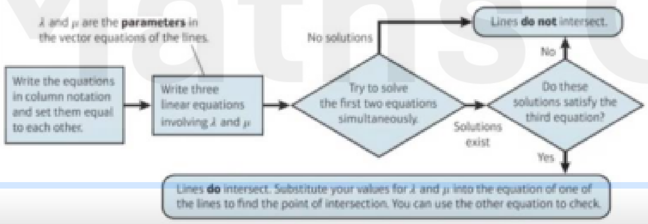
(numerator negated = position vector)

(denominator = direction vector)

$$\ell_1 : \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\ell_2 : \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

and seeing where the two lines intersect - if they intersect at a unique point, this means they lie in the same plane; following FLOWCHART (attached):



$$\begin{pmatrix} 1+2\lambda \\ -1-\lambda \\ 4+3\lambda \end{pmatrix} = \begin{pmatrix} 1+t \\ -t \\ 3+2t \end{pmatrix}$$

i component: $1+2\lambda = 1+t$

$\Rightarrow 2\lambda - t = 0$ - ①

j component: $-1-\lambda = -t$

$\Rightarrow t - \lambda = 1$ - ②

solve simultaneously (equtn solver/ by elimination)

P 6 1 1 7 7 A 0 2 4 3 2

Question 7 continued

①+②

$\Rightarrow \boxed{\lambda = 1}$

sub into ①

$2(1) - t = 0$

$\Rightarrow \boxed{t = 2}$

↳ checking these parameters also consistent with the 'k' component

LHS:

$4 + 3(1)$

$= 7$

RHS

$3 + 2(2)$

$= 7$

LHS = RHS ∴ consistent

↳ subbing into any of the general coordinates

$$\begin{pmatrix} 1 + 2(1) \\ -1 - 1 \\ 4 + 3(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$$

since lines intersect at a single point (consistent), must lie in the same plane //

WAY 2: algebraically - using Cartesian equtn for l_2

$$r = \begin{pmatrix} 1+t \\ -t \\ 3+2t \end{pmatrix}$$

$\Rightarrow x = 1 + t$

$y = -t$

$z = 3 + 2t$

and subbing into Cartesian equation:

$$\frac{1+t-1}{2} = \frac{-t+1}{-1} = \frac{3+2t-4}{3}$$

$$\Rightarrow \frac{t}{2} = \frac{-t+1}{-1} = \frac{-1+2t}{3}$$

splitting into

$$\frac{t}{2} = \frac{-t+1}{-1}$$

cross multiply

$-t = -2t + 2$

$\Rightarrow 2t - t = 2$

$\boxed{t = 2}$ → check if consistent with ②

Question 7 continued

$$\frac{-t+1}{-1} = \frac{-1+2t}{3}$$

cross multiply

$$-3t+3 = 1-2t$$

$$\Rightarrow \boxed{t=2}$$

$\therefore$ lines intersect at a single point where $t=2$ $\therefore$ lie in same plane //

Alternative method: finding equtns of planes for l_1 and l_2

need to find equtn of the plane that would CONTAIN the two lines: l_1 and l_2

↳ to do this, first need normal to both direction vectors

WAY 1: using dot product

let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the normal $\therefore$ using fact that normal is perpendicular

i.e dot product $=0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

multiply out:

$$2x-y+3z=0 \qquad x-y+2z=0$$

let $z=1$

$$2x-y=3 \text{ -①} \qquad x-y=-2 \text{ -②}$$

solve simultaneously – calc equtn solver/ by elimination

① – ②

$$\begin{aligned} 2x-y&=-3 \\ -\quad x-y&=-2 \\ \hline \boxed{x=-1} \end{aligned}$$

sub into ② for 'y':

$$(-1)-y=-2$$

$$\Rightarrow \boxed{y=1}$$

$\therefore$ normal to both

$$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

WAY 2: using cross product

to find vector perpendicular, just need to do the cross product of the two direction vectors

$$\begin{pmatrix}2\\-1\\3\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = i\begin{vmatrix}-1 & 3\\-1 & 2\end{vmatrix} - j\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} + k\begin{vmatrix}2 & -1\\1 & -1\end{vmatrix}$$

$$= i(1) - j(1) + k(-1)$$

$$= \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

---

now looking to get the 'd' from $r \cdot n = d$

...from ℓ_1:

$a \cdot n = d$

$$\begin{pmatrix}1\\-1\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = 1+1-4 = -2$$

$$\Rightarrow r \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = -2$$

...for ℓ_2:

$a \cdot n = d$

$$d = \begin{pmatrix}1\\0\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = 1-3 = -2$$

$$\Rightarrow r \cdot \begin{pmatrix}1\\-1\\-1\end{pmatrix} = -2$$

$\Rightarrow$ $\boxed{\text{lines lie on same plane}}$

(b) finding the vector parametric form $\Rightarrow$ general form is $r = a + \lambda b + \mu c$

↳ from part (a) know that the two lines ℓ_1 and ℓ_2 lie on same plane ; can pick any of the two position vectors of ℓ_1 and ℓ_2 as 'a' and then the two direction vectors for 'b' and 'c' (two non-parallel vectors)

$$\Rightarrow \pi : r = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

or

$$r = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

(c) to find the acute angle between ℓ_1 and ℓ_2 - follow the formula:

$$\cos\theta = \frac{|b_1 \cdot b_2|}{|b_1||b_2|}$$

$$\cos\theta = \frac{\begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix}}{\sqrt{(2)^2+(-1)^2+(3)^2}\sqrt{(1)^2+(-1)^2+(2)^2}}$$

$$= \frac{2+1+6}{\sqrt{14}\sqrt{6}} = \frac{9}{\sqrt{14}\sqrt{6}} = \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{9}{\sqrt{14}\sqrt{6}}\right)$$

$$= 10.893...$$

$$= \boxed{11^\circ}$$

Question 7 continued

(Total for Question 7 is 7 marks)

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{dw}{dt} = \frac{5}{2}(w - s) \quad -①$$
$$\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t} \quad -②$$

(a) Show that

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years.

(6)

(c) Find a general solution for the number of signal crayfish at time t years.

(2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places.

(6)

(e) Suggest a limitation of the model.

(1)

(a) notice we are asked to get an equation only in terms of 'w' and its derivatives

∴ differentiating ① wrt 't'

$$\frac{d^2w}{dt^2} = \frac{5}{2}\left(\frac{dw}{dt} - \frac{ds}{dt}\right) \quad -③$$

and subbing ② into ③ to get rid of $\frac{ds}{dt}$

$$\frac{d^2w}{dt^2} = \frac{5}{2}\left(\frac{dw}{dt} - \left(\frac{2}{5}w - 90e^{-t}\right)\right)$$

×2

$$2\frac{d^2w}{dt^2} = 5\left(\frac{dw}{dt} - \frac{2}{5}w + 90e^{-t}\right)$$

multiply 5 in

$$2\frac{d^2w}{dt^2} = 5\frac{dw}{dt} - 2w + 450e^{-t}$$

DO NOT WRITE IN THIS AREA

P 6 1 1 7 7 A 0 2 8 3 2

Question 8 continued

take u terms to LHS

$$\boxed{2\frac{d^2u}{dt^2} - 5\frac{du}{dt} + 2u = 450e^{-t}}$$

(b) asked to solve non-homogenous 2ODE

$\boxed{\text{A.E}}$: $2m^2 - 5m + 2 = 0$

↳ calc equtn solver/quadratic formula

$$m = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm \sqrt{9}}{4}$$

...+ve:

$$m = \frac{5+3}{4}$$
$$= 2$$

...-ve:

$$m = \frac{5-3}{4}$$
$$= \frac{2}{4} = \frac{1}{2}$$

hints at 2 real solutions $\therefore$ using general equation : $u = Ae^{\alpha t} + Be^{\beta t}$

$\Rightarrow$ $\boxed{\text{C.F}}$: $u = Ae^{0.5t} + Be^{2t}$

now P.I from table:

try $u = \lambda e^{-t}$

$$\frac{du}{dt} = -\lambda e^{-t}$$

$$\frac{d^2u}{dt^2} = \lambda e^{-t}$$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

WARNING!
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form

sub into 2ODE

$$2(\lambda e^{-t}) - 5(-\lambda e^{-t}) + 2(\lambda e^{-t}) = 450e^{-t}$$

expanding out :

$$2\lambda e^{-t} + 5\lambda e^{-t} + 2\lambda e^{-t} = 450e^{-t}$$

$$\Rightarrow 9\lambda = 450 \quad \div 9$$

$$\div 9 \Rightarrow \lambda = 50$$

Question 8 continued

G.S = C.F + P.I

$$\boxed{\therefore w = Ae^{0.5t} + Be^{2t} + 50e^{-t}}$$

(c) for no. of s want to sub in to ① ∴ differentiate part (b)

$$\frac{dw}{dt} = \frac{1}{2}Ae^{0.5t} + 2Be^{2t} - 50e^{-t}$$

and sub into ①

$$\frac{1}{2}Ae^{0.5t} + 2Be^{2t} - 50e^{-t} = \frac{5}{2}\left(Ae^{0.5t} + Be^{2t} + 50e^{-t} - s\right)$$

expand

$$\frac{1}{2}Ae^{0.5t} + 2Be^{2t} - 50e^{-t} = \frac{5}{2}Ae^{0.5t} + \frac{5}{2}Be^{2t} + 125e^{-t} - \frac{5}{2}s$$

take s to LHS and rest to RHS

$$\frac{5}{2}s = 2e^{0.5t} + \frac{1}{2}Be^{2t} + 175e^{-t}$$

$$\boxed{\Rightarrow s = \frac{4}{5}Ae^{0.5t} + \frac{1}{5}Be^{2t} + 70e^{-t}}$$

(d) "will have died out" suggests T where $w=0$

↳ subbing in initial conditions into G.S from part(b) and s from part (c)

when $t=0$, $w=65$

$$65 = Ae^{0.5(0)} + Be^{2(0)} + 50e^{-0}$$

$$\Rightarrow 65 = A + B + 50$$

$$\Rightarrow A + B = 15 \quad -①$$

when $t=0$, $s=85$

$$85 = \frac{4}{5}Ae^{0.5(0)} + \frac{1}{5}Be^{2(0)} + 70e^{-0}$$

$$\Rightarrow 85 = \frac{4}{5}A + \frac{1}{5}B + 70$$

$$\Rightarrow \frac{4}{5}A + \frac{1}{5}B = 15 \quad \times 5$$

$$4A + B = 75 \quad -②$$

solving this using calc equtn solver/by elim.

$$② - ① \quad 3A = 60$$

$$\div 3 \qquad \div 3$$

$$A = 20$$

sub into ②

$$80 + B = 75$$

$$\Rightarrow B = -5$$

$$\therefore w = 20e^{0.5t} - 5e^{2t} + 50e^{-t}$$

$$s = 16e^{0.5t} - 5e^{2t} + 70e^{-t}$$

need 't' where $w=0$

$$20e^{0.5t} - 5e^{2t} + 50e^{-t} = 0$$

$$\times e^{t} \qquad \times e^{t}$$

Question 8 continued

$$20e^{1.5t} - 5e^{3t} + 50 = 0$$

$$\Rightarrow 5e^{3t} - 20e^{1.5t} - 50 = 0$$

$\div 5$ $\qquad \div 5$

$$e^{3t} - 4e^{1.5t} - 10 = 0$$

notice this is a quadratic in $e^{1.5t}$, so using substitution : $y = e^{1.5t}$

$$y^2 - 4y - 10 = 0$$

calc equtn solver/quadratic formula

$$y = \frac{4 \pm \sqrt{16 - 4(1)(-10)}}{2}$$

$$e^{1.5t} = \frac{4 \pm \sqrt{56}}{2}$$

taking logs of both sides

$$1.5t = \ln\left(\frac{4 \pm \sqrt{56}}{2}\right)$$

but can't take logs of -ves

$$\therefore 1.5t = \ln\left(\frac{4 + \sqrt{56}}{2}\right)$$

$\div 1.5 \qquad \div 1.5$

$$t = \frac{2}{3}\ln\left(\frac{4 + \sqrt{56}}{2}\right)$$

$$= \boxed{1.165}$$

e) possible limitations:

- either population becomes -ve which isn't possible
- when white-clawed crayfish die out, model becomes invalid

Question 8 continued

(Total for Question 8 is 18 marks)

TOTAL FOR PAPER IS 75 MARKS